

**Summary of Research**  
**NASA Grant NAG-1-1850**  
**Stress Recovery and Error Estimation for 3-D Shell Structures**

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**Summary**

The  $C^1$ -continuous stress fields obtained from finite element analyses are in general lower-order accurate than are the corresponding displacement fields. Much effort has focussed on increasing their accuracy and/or their continuity, both for improved stress prediction and especially error estimation. A previous project developed a penalized, discrete least squares variational procedure that increases the accuracy and continuity of the stress field. The variational problem is solved by a post-processing, 'finite-element-type' analysis to recover a smooth, more accurate,  $C^1$ -continuous stress field given the 'raw' finite element stresses. This analysis has been named the SEA/PDLS. The recovered stress field can be used in *a posteriori* error estimators, such as the Zienkiewicz-Zhu error estimator or equilibrium error estimators. The procedure was well-developed for the two-dimensional (plane) case involving low-order finite elements. It has been demonstrated that, if optimal finite element stresses are used for the post-processing, the recovered stress field is globally superconvergent. Extension of this work to three dimensional solids is straightforward.

The extension of the smoothing procedure to curved, 3-dimensional surfaces modelled by shell elements is much more difficult than extending the procedure to a 3-dimensional solid. This extension has been a primary focus of this project. One major difficulty in the extension is that the shell finite element mesh only approximates geometrically the true surface. In addition, a smoothing mesh will also represent only approximately the true surface. As a result, in general there will be two approximate, and distinct, surfaces, defined by the finite element and smoothing element meshes, as well as the true geometric surface. For smoothing to occur, a mapping procedure between the two meshes is required. Another major difficulty arises because the SEA methodology recovers (nearly) continuous gradients as well as continuous stresses. However, on a curved surface the meaning of the stress gradient is not well-defined in 3-D space, but rather it is only well-defined on the 2-D parametric space which describes the surface. Therefore, the gradients should only be defined relative to the surface.

As technology advances, meshes are based more and more on geometric descriptions of the actual surfaces. Within 10 years, it is likely that the geometric representation of the structure will replace the finite element mesh as the object with which the computer analyst interacts for a majority of problems. Hence, the work herein is restricted to these cases.

An analytical surface in 3-D space can be described in a 2-D parametric coordinate space. Practical structural surfaces can be built-up of such surfaces. The recovery strategy developed in this project involves mapping the finite element results to the 2-D parametric space which describes the geometry, and then smoothing is carried out in the parametric space using the 2-D SEA/PDLS methodology developed previously. The procedure has been presented and demon-

strated in publication 1. The recovered stresses were used in the Zienkiewicz-Zhu *a posteriori* error estimator, and the estimated errors were used to demonstrate the performance of SEA-recovered stresses in automated adaptive mesh refinement of shell structures. The numerical results are encouraging. Further testing involving more complex, practical structures is necessary.

As mentioned previously, SEA/PDLS can recover a superconvergent stress field if optimal finite element stresses are used in the analysis. However, many shell elements are formulated such that the location of optimal stress points are unknown or do not exist. Therefore, a secondary effort in this project was to extend the MIN3S triangular shell element, based on first-order shear deformation theory, to a quadrilateral for which optimal stresses could be obtained. These optimal stresses can then be used for the smoothing analysis to recover a superconvergent stress field. The shell element which resulted from this work, MIN4T, was constructed of four MIN3 elements. The degrees-of-freedom associated with the interior node were eliminated via explicit, analytical constraint equations based on the Kirchhoff constraints of zero transverse shear. The result is a four-node, 24 DOF, shear deformable shell element with one optimal stress point. A major presentation of this work is found in publication 2.

A secondary effort of the latter phase of this project has been to extend the SEA/PDLS methodology to higher-order elements. As developed previously, the methodology had been developed for low-order finite elements (linear and quadratic variation of displacements). This limitation is based on the quadratic interpolation functions for the smoothing element. SEA/PDLS functions optimally when the interpolation functions for the smoothing element is one degree higher than the consistent finite element stress field. The smoothing element can be considered to belong to a class of constrained, anisoparametric elements. A general methodology to develop interpolation functions for higher-order, constrained, anisoparametric elements has been developed, and the interpolation functions for a cubic element have been derived explicitly. The work in this area is continuing. Our intention is to use these interpolation functions both for a higher-order smoothing element and a higher-order anisoparametric plate element.

The smoothing procedure has been implemented in the computer program MANOA, an open development environment for Matrix And Numerical Oriented Analysis. The shell element MIN4T is also implemented in MANOA. MANOA is available at <http://www.eng.hawaii.edu/~riggs/home.html>.

The following publications are based on work performed under this grant.

1. Yazdani, A. A., Riggs, H. R. and Tessler, A. (2000), 'Stress Recovery and Error Estimation for Shell Structures,' *International Journal for Numerical Methods in Engineering*, Vol. 47, pp. 1825-1840.

### **Abstract**

The Penalized Discrete Least-Squares (PDLS) stress recovery (smoothing) technique developed for two-dimensional linear elliptic problems is adapted here to three-dimensional shell structures. The surfaces are restricted to those which have a 2-D parametric representation, or which can be built-up of such surfaces. The proposed strategy involves mapping the finite element results to the 2-D parametric space which describes the geometry, and smoothing is carried out in the parametric space using the PDLS-based Smoothing Element Analysis (SEA). Numerical results for two well-known shell problems are presented to illustrate the performance of SEA/PDLS for these problems. The recovered

stresses are used in the Zienkiewicz-Zhu *a posteriori* error estimator. The estimated errors are used to demonstrate the performance of SEA-recovered stresses in automated adaptive mesh refinement of shell structures. The numerical results are encouraging. Further testing involving more complex, practical structures is necessary.

2. Liu, J., Riggs, H. R. and Tessler, A. (2000), 'A 4-Node, Shear-Deformable Flat Shell Element Developed via Explicit Kirchhoff Constraints,' *International Journal for Numerical Methods in Engineering*, to appear.

#### **Abstract**

An efficient, four-node quadrilateral shell element is formulated using a linear, first-order shear deformation theory. The bending part of the formulation is constructed from a cross-diagonal assembly of four three-node anisoparametric triangular plate elements, referred to as MIN3. Closed-form constraint equations, which arise from the Kirchhoff constraints in the thin-plate limit, are derived and used to eliminate the degrees-of-freedom associated with the 'internal' node of the cross-diagonal assembly. The membrane displacement field employs an Allman-type, drilling degrees-of-freedom formulation. The result is a displacement-based, fully-integrated, four-node quadrilateral element, MIN4T, possessing six degrees-of-freedom at each node. Results for a set of validation plate problems demonstrate that the four-node MIN4T has similar robustness and accuracy characteristics as the original cross-diagonal assembly of MIN3 elements involving five nodes. The element performs well in both moderately thick and thin regimes, and it is free of shear locking. Shell validation results demonstrate superior performance of MIN4T over MIN3, possibly as a result of its higher-order interpolation of the membrane displacements. It is also noted that the bending formulation of MIN4T is kinematically compatible with the existing anisoparametric elements of the same order of approximation, which include a two-node Timoshenko beam element and a three-node plate element, MIN3.

3. Tessler, A., Riggs, H. R. and Dambach, M. (1999), 'A Novel Four-Node Quadrilateral Smoothing Element for Stress Enhancement and Error Estimation,' *International Journal for Numerical Methods in Engineering*, Vol. 44, No. 10, pp. 1527-1543.

#### **Abstract**

A four-node, quadrilateral smoothing element is developed based upon a penalized-discrete-least-squares variational formulation. The smoothing methodology recovers  $C^1$ -continuous stresses, thus enabling effective *a posteriori* error estimation and automatic adaptive mesh refinement. The element formulation is originated with a five-node macro-element configuration consisting of four triangular anisoparametric smoothing elements in a cross-diagonal pattern. This element pattern enables a convenient closed-form solution for the degrees of freedom of the interior node, resulting from enforcing explicitly a set of natural edge-wise penalty constraints. The degree-of-freedom reduction scheme leads to a very efficient formulation of a four-node quadrilateral smoothing element without any compromise in robustness and accuracy of the smoothing analysis. The application examples include stress recovery and error estimation in adaptive mesh refinement solutions for

an elasticity problem and an aerospace structural component.

4. Tessler, A., Riggs, H. R. and Dambach, M. (1998), 'A Novel Four-Node Quadrilateral Smoothing Element for Stress Enhancement and Error Estimation,' *Proceedings, 39th Structures, Structural Dynamics, and Materials Conference*, Long Beach, CA, AIAA-98-1713.

#### **Abstract**

A four-node, quadrilateral smoothing element is developed based upon a penalized-discrete-least-squares variational formulation. The smoothing methodology recovers  $C^1$ -continuous stresses, thus enabling effective *a posteriori* error estimation and automatic adaptive mesh refinement. The element formulation is originated with a five-node macro-element configuration consisting of four triangular anisoparametric smoothing elements in a cross-diagonal pattern. This element pattern enables a convenient closed-form solution for the degrees of freedom of the interior node, resulting from enforcing explicitly a set of natural edge-wise penalty constraints. The degree-of-freedom reduction scheme leads to a very efficient formulation of a four-node quadrilateral smoothing element without any compromise in robustness and accuracy of the smoothing analysis. The application examples include stress recovery and error estimation in adaptive mesh refinement solutions for an elasticity problem and an aerospace structural component.

5. Liu, Y. J., Riggs, H. R. and Tessler, A. (1998), 'A 4-Node Anisoparametric Mindlin Plate Element Based on the Tessler 3-Node Element,' Report UHM/CE/98-01, University of Hawaii, 81 pp.

#### **Abstract**

The formulation for an efficient, linear, 4-node, 24-degree-of-freedom quadrilateral flat shell element based on uncoupled bending and membrane behaviors is presented. The shell is formulated by combining a 4-node, 12 DOF quadrilateral plate element and a 4-node 12 DOF quadrilateral membrane element. The plate element, MIN4T, is developed herein using the successful 3-node, 9 DOF triangular plate element MIN3 as a basis. The membrane element is based on an existing formulation for a 4-node quadrilateral element. The shell element formulation has been evaluated numerically by considering the uncoupled problems. That is, a 4-node, 12 DOF plate element has been developed to evaluate the bending behavior and a 4-node, 12 DOF membrane element has been developed to evaluate the in-plane behavior. Both bending and membrane elements are tested for robustness and accuracy. Results of the bending element for both isotropic and orthotropic materials show that MIN4T is as robust and accurate as the original triangular element. The numerical studies illustrate that the membrane element with drilling degrees of freedom exhibits excellent accuracy.

6. Riggs, H. R., Liu, J., and Tessler, A. (1997), 'An Edge-Constrained, Four-Node Quadrilateral Plate Element Based on First-Order Shear Deformation Theory,' *Proceedings, Fourth U.S. National Congress on Computational Mechanics*, San Francisco, p. 258 (abstract only).

#### **Abstract**

The development of low-order, shear-deformable Reissner-Mindlin plate elements that are both robust and efficient has proven to be a challenge. The primary difficulty is the tendency of many formulations to experience significant shear locking in the thin-plate regime. Tessler and Hughes proposed an exactly integrated, three-node triangular element (MIN3) which uses anisoparametric interpolation functions and a shear correction factor to eliminate shear locking, even for very thin plates. This element has proven to be very robust, and because it is low order it is computationally efficient. The present paper develops and demonstrates a four-node quadrilateral element for which MIN3 serves as a building block. A quadrilateral formed by a patch of 4 MIN3 elements in a cross diagonal pattern involves 5 nodes; explicit Kirchhoff constraints can be used to eliminate analytically the degrees-of-freedom associated with the interior node. The paper presents in detail the constraints used to formulate the resulting four-node element. The element is tested for robustness and accuracy, with special emphases on shear locking and a comparison with MIN3. Applications for both isotropic plates and uni-directional composite laminates are considered. The results show that MIN4T is robust and computationally efficient, and its performance compares favorably to that of MIN3. One of the element's advantages over the triangular element is that for a mesh of  $N \times N$  quadrilaterals, the mesh with MIN4T elements would have approximately 1/2 the DOFs than a corresponding cross-diagonal mesh constructed with MIN3 elements.